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Constructions of perfect Mendelsohn designs

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Abstract

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Let n and k be positive integers. An $(n, k, 1)$-Mendelsohn design is an ordered pair $(V, \mathscr{C})$ where V is the vertex set of D_n, the complete directed graph on n vertices, and $\mathscr{C}$ is a set of directed cycles (called blocks) of length k which form an arc-disjoint decomposition of D_n. An $(n, k, 1)$-Mendelsohn design is called a *perfect* design and denoted briefly by $(n, k, 1)$-PMD if for any r, $1 \leq r \leq k-1$, and for each $(x, y) \in V \times V$ there is exactly one cycle $c \in \mathscr{C}$ in which the (directed) distance along c from x to y is r. A necessary condition for the existence of an $(n, k, 1)$-PMD is $n(n-1) \equiv 0 \pmod k$. In this paper we shall describe some new techniques used in the construction of PMD's, including constructions of the product type. As an application, we show that the necessary condition for the existence of an $(n, 5, 1)$-PMD is also sufficient, except for $n = 6$ and with at most 21 possible exceptions of n of which 286 is the largest.

1. Introduction

Let D_n be the complete directed graph on n vertices. An $(n, k, 1)$-*Mendelsohn design* is an ordered pair $(V, \mathscr{C})$ where V is the vertex set of D_n and $\mathscr{C}$ is a set of directed cycles (called *blocks*) of length k which form an arc-disjoint decomposition of D_n. Denote by $(v_0, v_1, \ldots, v_{k-1})$ the directed k-cycle consisting of the arcs (v_i, v_{i+1}) for $0 \leq i \leq k-2$ and (v_{k-1}, v_0).

An $(n, k, 1)$-Mendelsohn design $(V, \mathscr{C})$ is r-*perfect* if for each $(x, y) \in V \times V$ there is exactly one cycle $c \in \mathscr{C}$ in which the (directed) distance along c from x to

y is r. So, by definition, every Mendelsohn design is 1-perfect. An $(n, k, 1)$-Mendelsohn design is *perfect* if it is r-perfect for $1 \leqslant r \leqslant k-1$, denoted briefly by $(n, k, 1)$-PMD.

The concept of a PMD was introduced by N.S. Mendelsohn [12] under the name of perfect cyclic design and further studied in a subsequent paper [3]. The terminology of Mendelsohn design was first used in its general form by Hsu and Keedwell [9].

A simple counting argument shows that the number of blocks of a $(n, k, 1)$-PMD is $n(n-1)/k$. It follows that a necessary condition for the existence of an $(n, k, 1)$-PMD is

$$n(n-1) \equiv 0 \pmod{k}. \tag{1.1}$$

For the sufficiency, constructions using finite fields provide us with the following two theorems (see, for example, [3, 9, 12]).

Theorem 1.1. *Let p be an odd prime and $r \geqslant 1$, then there exists a $(p^r, p, 1)$-PMD.*

Theorem 1.2. *Let $n = p^r$ be any prime power and $k > 2$ be such that k is a divisor of $n-1$, then there exists an $(n, k, 1)$-PMD.*

Some 'asymptotic' results are also provided in [3] and [12].

Theorem 1.3. *An $(n, k, 1)$-PMD exists for all sufficiently large n with $k \geqslant 3$ and $n \equiv 1 \pmod{k}$.*

Theorem 1.4. *An $(n, k, 1)$-PMD exists with $n(n-1) \equiv 0 \pmod{k}$ for the case when k is an odd prime and n is sufficiently large.*

We remark that the term 'sufficiently large' in Theorems 1.3 and 1.4 is unspecified and the problem of finding a concrete bound for n in both cases remains to be solved. For small k, the following result is contained in [11].

Theorem 1.5. *An $(n, 3, 1)$-PMD exists if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$.*

The following result is shown in [2, 4, 23].

Theorem 1.6. *An $(n, 4, 1)$-PMD exists if and only if $n \equiv 0$ or $1 \pmod 4$, with the exception of $n = 4$, 8 and the possible exception of $n = 12$.*

The purpose of this paper is to describe some new techniques used in the construction of PMDs. As an application, we show that the necessary condition for the existence of an $(n, 5, 1)$-PMD, namely, $n \equiv 0$ or $1 \pmod 5$, is also sufficient, except for $n = 6$ and the possible exception of $n \in \{10, 15, 20, 26, 30, 36, 46, 50, 56, 66, 86, 90, 110, 126, 130, 140, 146, 186, 206, 246, 286\}$.

2. Constructions from Steiner pentagon systems

A *Steiner pentagon system* (SPS) is a pair (K_n, P) where K_n is the complete undirected graph (based on the set V), P is a collection of pentagons in K_n such that each edge of K_n belongs to exactly one pentagon of P, and each pair of distinct elements of V are joined by a path of length 2 in exactly one pentagon of P. The number n is called the order of the SPS (K_n, P) and, of course, $|P| = n(n-1)/10$.

It has been proved in [10] that for all $n \equiv 1$ or $5 \pmod{10}$, except 15, there exist SPSs of order n. An observation shows that the existence of an SPS of order n implies the existence of an $(n, 5, 1)$-PMD. By assigning to each pentagon $\langle a, b, c, d, e\rangle$ of the SPS of order n, the two directed cycles (a, b, c, d, e) and (a, e, d, c, b), these directed cycles form not only a partition of the arcs for the complete directed graph D_n, but also a perfect Mendelsohn design. We then have the following.

Theorem 2.1. *An* $(n, 5, 1)$-PMD *exists for any integer* $n \geqslant 5$ *and* $n \equiv 1$ *or* $5 \pmod{10}$, *except possibly* $n = 15$.

3. The kn and $kn+1$ constructions

We denote by $K_{n_1,n_2,\ldots,n_h}$ the complete multipartite directed graph with vertex set $X = \bigcup_{1\leqslant i\leqslant h} X_i$, where X_i are disjoint sets with $|X_i| = n_i$ and where two elements x and y from different sets X_i and X_j are joined by exactly two arcs (x, y) and (y, x).

An $(n, k, 1)$-*holey Mendelsohn design* is an ordered pair $(X, \mathscr{C})$ where $\mathscr{C}$ is a set of directed cycles of length k which form an arc-disjoint decomposition of $K_{n_1,n_2,\ldots,n_h}$. Denote by $(x_0, x_1, \ldots, x_{k-1})$ the directed k-cycle consisting of the arcs (x_i, x_{i+1}) for $0 \leqslant i \leqslant k-2$ and (x_{k-1}, x_0).

An $(n, k, 1)$-holey Mendelsohn design $(X, \mathscr{C})$ is *r-perfect* if for any vertices x and y belonging to different sets X_i and X_j there is exactly one cycle $c \in \mathscr{C}$ such that the (directed) distance along c from x to y is r. An $(n, k, 1)$-holey Mendelsohn design is *perfect* if it is r-perfect for $1 \leqslant r \leqslant k-1$. A perfect $(n, k, 1)$-holey Mendelsohn design $(X, \mathscr{C})$ is denoted briefly by $(n, k, 1)$-HPMD. Each X_i is called a *hole* of the design. The vector $(n_1, n_2, \ldots, n_h)$ or its product form $n_1 n_2 \cdots n_h$ is called the *type* of the holey design. We also use the notation $1^{u_1}2^{u_2}\cdots$ to describe the type, where there are u_i holes of size i, $i \geqslant 1$.

An $(n, k, 1)$-HPMD of type $1^{n-m}m^1$ is called an *incomplete perfect Mendelsohn design*, denoted by $(n, m, k, 1)$-IPMD. An $(n, k, 1)$-PMD is nothing else but an $(n, k, 1)$-HPMD of type 1^n. The following lemmas are obvious by filling in holes.

Lemma 3.1. *If there exists an* $(n, m, k, 1)$-IPMD *and an* $(m, k, 1)$-PMD, *then there exists an* $(n, k, 1)$-PMD.

Proof. Suppose that $(X, \mathscr{C})$ and $(Y, \mathscr{C}')$ are the given IPMD and PMD where $Y \subset X$. Then $(X, \mathscr{C} \cup \mathscr{C}')$ is the required $(n, k, 1)$-PMD. $\square$

Lemma 3.2. *If there exists an $(n, k, 1)$-HPMD of type $(n_1, n_2, \ldots, n_h)$ and an $(n_i + m, k, 1)$-IPMD for each $i, 2 \leqslant i \leqslant h$, then there exists an $(n + m, n_1 + m, k, 1)$-IPMD.*

Proof. Let $(X, \mathscr{C})$ be the given HPMD. X is partitioned into $X_1, X_2, \ldots, X_h$, $|Y| = m$, $Y \cap X = \emptyset$. Let $(X_i \cup Y, \mathscr{C}_i)$ be the given IPMD for each $i, 2 \leqslant i \leqslant h$. Then $(X \cup Y, (\bigcup_{2 \leqslant i \leqslant h} \mathscr{C}_i) \cup \mathscr{C})$ is the required IPMD with a hole on $X_1 \cup Y$. $\square$

To describe the constructions for HPMD and PMD we need the concept of a k-difference sequence and something about orthogonal Latin squares.

Let k be an odd integer. Denote by $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ the ring of residue classes modulo k. Let $S = (s_0, s_1, \ldots, s_{k-1})$, $s_i \in \mathbb{Z}_k$. If for any $r \in \mathbb{Z}_k \setminus \{0\}$,

$$\{s_{i+r} - s_i : i \in \mathbb{Z}_k\} = \mathbb{Z}_k,$$

where the sum $i + r$ is calculated in $\mathbb{Z}_k$, we shall call S a *k-difference sequence.* For odd prime k it is easy to verify that $(0^2, 1^2, \ldots, (k-1)^2)$ is a k-difference sequence.

Let $P = \{X_1, X_2, \ldots, X_h\}$ be a partition of an n-set X. A *holey Latin square,* having partition P, is an $n \times n$ array L, indexed by X, satisfying the following properties:

(1) a cell of L either contains an element of X or is empty;

(2) the subarray indexed by $X_i \times X_i$ is empty, for $1 \leqslant i \leqslant h$ (these subarrays are called *holes*);

(3) the elements occurring in row (or column) $x \in X_i$ of L are precisely those in $X \setminus X_i$.

The *type* of L is the vector $(n_1, n_2, \ldots, n_h)$ where $n_i = |X_i|$. We also use the notation $1^{u_1} 2^{u_2} \cdots$ to describe the type, where there are u_i holes of size i, $i \geqslant 1$.

Suppose $L_1, L_2, \ldots, L_t$ are holey Latin squares having partition P. If for any $1 \leqslant i < j \leqslant t$ the superposition of L_i and L_j yields every ordered pair in $X^2 \setminus \bigcup_{i=1}^{h} X_i^2$, we call them *$t$ holey mutually orthogonal Latin squares* of order n with type $(n_1, n_2, \ldots, n_h)$, denoted by t HMOLS(n).

It is well known (see [8]) that a Latin square L based on set X is a multiplication table of some quasigroup $(X, \circ)$, and vice versa. Accordingly, we have quasigroups with holes and their orthogonality.

Theorem 3.3. *Let k be an odd prime. Suppose that there exist $k-2$ HMOLS(n) of type $(n_1, n_2, \ldots, n_h)$. Then there exists a $(kn, k, 1)$-HPMD of type $(kn_1, kn_2, \ldots, kn_h)$.*

Proof. Let the $k-2$ HMOLS(n) be based on the set X with disjoint holes $X_1, X_2, \ldots, X_h$ where $|X_i| = n_i$. The corresponding quasigroups will be denoted by $(X, \circ_t)$, $t = 0, 1, \ldots, k-3$. Let $S = (s_0, s_1, \ldots, s_{k-1})$ be a k-difference sequence on $\mathbb{Z}_k$. We shall construct the required HPMD by $(X \times \mathbb{Z}_k, \mathscr{C})$ where $\mathscr{C}$ consists of the following directed cycles:

$$((x \circ_0 y, s_0 + g), (x \circ_1 y, s_1 + g), \ldots, (x \circ_{k-3} y, s_{k-3} + g), (x, s_{k-2} + g), (y, s_{k-1} + g)),$$

where x and y belong to different sets X_i and X_j, $g \in \mathbb{Z}_k$.

For any two elements (s, a) and (t, b) belonging to different holes $X_i \times \mathbb{Z}_k$ and $X_j \times \mathbb{Z}_k$, s and t belong to different holes X_i and X_j. To find the cycle along which the distance from (s, a) to (t, b) is r, we determine the unique i from the definition of k-difference sequence such that

$$s_{i+r} - s_i \equiv b - a \pmod{k}.$$

Then by the orthogonality of the quasigroups there is a unique x and y such that $x \circ_i y = s$, $x = s$ or $y = s$ according to whether $i < k-2$, $i = k-2$ or $i = k-1$ respectively and such that $x \circ_{i+r} y = t$, $x = t$ or $y = t$ according to whether $i + r < k-2$, $i + r = k-2$ or $i + r = k-1$ respectively. Once i, x, y are determined we need only determine g to get the cycle, this can be done by taking $g = a - s_i$.

On the other hand, (s, a) and (t, b) cannot appear in two cycles with the same distance r. Otherwise, there will be i_1, g_1, x_1 and y_1 such that

$$\begin{aligned}
&x \circ_i y = x_1 \circ_{i_1} y_1 = s,\\
&x \circ_{i+r} y = x_1 \circ_{i_1+r} y_1 = t,\\
&s_i + g = s_{i_1} + g_1 = a,\\
&s_{i+r} + g = s_{i_1+r} + g_1 = b.
\end{aligned}$$

The definition of k-difference sequence guarantees that $i = i_1$ and $g = g_1$. The orthogonality of quasigroups $(X, \circ_i)$ and $(X, \circ_{i+r})$ implies that $(x, y) = (x_1, y_1)$. This is a contradiction and the proof is complete. □

From Lemma 3.1 and Lemma 3.2 we have the following corollary.

Corollary 3.4. *Let k be an odd prime. Suppose that there exist $k-2$ HMOLS(n) of type $(n_1, n_2, \ldots, n_h)$. Then we have the following constructions.*

(i) (***kn*-construction**). *A $(kn, k, 1)$-PMD exists if a $(kn_i, k, 1)$-PMD exists for all i, $1 \leq i \leq h$; and*

(ii) (***kn* + 1-construction**). *A $(kn+1, k, 1)$-PMD exists if a $(kn_i + 1, k, 1)$-PMD exists for all i, $1 \leq i \leq h$.*

More specifically, we have the following for the case $k = 5$.

Corollary 3.5 (5*n*-construction). *If there exist 3 HMOLS(n) of type $(n_1, n_2, \ldots, n_h)$, and a $(5n_i, 5, 1)$-PMD for $1 \leq i \leq h$, then there exists a $(5n, 5, 1)$-PMD.*

Corollary 3.6 **($5n+1$-construction).** *If there exist* 3 HMOLS(n) *of type* $(n_1, n_2, \ldots, n_h)$, *and a* $(5n_i+1, 5, 1)$-PMD *for* $1 \leq i \leq h$, *then there exists a* $(5n+1, 5, 1)$-PMD.

4. Construction of $(n, 5, 1)$-PMDs, $n \equiv 0 \pmod{10}$

It is obvious from Theorem 1.1 that a $(5, 5, 1)$-PMD exists. By the $5n$-construction we can obtain some $(5n, 5, 1)$-PMD if we have 3 HMOLS(n) of type 1^n. The case when n is even will give the required PMD for $n \equiv 0 \pmod{10}$. For this, we need the concept of an idempotent Latin square.

An *idempotent Latin square* of order n is a Latin square of order n based on the set $\mathbb{Z}_n$ with entry i in the cell (i, i) for $0 \leq i \leq n-1$. If we have t mutually orthogonal idempotent Latin squares of order n, briefly t idempotent MOLS(n), and if we delete the entries on the main diagonal for each square, we obtain t HMOLS(n) of type 1^n.

A *transversal* of a Latin square of order n is a set of n cells on different rows, different columns and with different entries. An idempotent Latin square is clearly such a square. On the other hand, by permuting row indices, column indices and entries, we can obtain an idempotent Latin square from some Latin square with a transversal. Moreover, the existence of t MOLS(n) with a common transversal is equivalent to the existence of t idempotent MOLS(n) and hence equivalent to the existence of t HMOLS(n) of type 1^n.

It is well known that a Latin square with an orthogonal mate has disjoint transversals, each of which comes from cells with the same entry in the orthogonal mate. Therefore, the existence of $t+1$ MOLS(n) implies the existence of t MOLS(n) with n disjoint common transversals and hence the existence of t HMOLS(n) of type 1^n. From [5, 7, 14, 18], we have the following known result on 4 MOLS(n).

Lemma 4.1. *There exist* 4 MOLS(n) *for any positive integer* $n \notin \{2, 3, 4, 6, 10, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 52\}$.

Lemma 4.2. *There exist* 3 *idempotent* MOLS(n) *for* $n = 14, 30, 34, 38, 42$.

Proof. Todorov provided in [17] 3 MOLS(14) which in fact have a common transversal. Wilson in [20] gave 3 MOLS(30) which are readily checked to have five common transversals. Wang gave (see [5, p. 404]) 3 MOLS(n) for $n = 34, 38, 42$ with subsquares of order 8, these 3 MOLS have respectively two, six, eight common transversals. This completes the proof. □

To prove the existence of 3 MOLS(n) of type 1^n for $n = 44, 52$, we use the following theorem of Wilson [21] under the terminology of transversal designs.

Theorem 4.3. *Suppose* $(X, \mathcal{G}, \mathcal{B})$ *is a* $\mathrm{TD}(k+l, t)$ *with groups* $G_1, G_2, \ldots, G_k, H_1, H_2, \ldots, H_l$. *Let* S *be any given subset of* $H_1 \cup H_2 \cup \cdots \cup H_l$ *and* m *be any given nonnegative integer. Suppose there exist transversal designs of the following kinds:*

(i) *for each* $i = 1, 2, \ldots, l$, *a* $\mathrm{TD}(k, h_i)$, *where* $h_i = |S \cap H_i|$;

(ii) *for each block* $A \in \mathcal{B}$, *a* $\mathrm{TD}(k, m + u_A)$ *which contains a set of* u_A *pairwise disjoint blocks, where* $u_A = |S \cap A|$. *Then there exists a* $\mathrm{TD}(k, mt + s)$, *where* $s = |S|$.

Applying Theorem 4.3 with $l = 1$, $m = 4$, $s = 8$ and $t = 9, 11$, we obtain TD(5, 44) and TD(5, 52), i.e., 3 MOLS(44) and 3 MOLS(52). Examining the proof of Theorem 4.3 in detail (see [19]), one finds that the above TDs all have a parallel class of blocks so that the corresponding 3 MOLS have common transversals. We then have the following.

Lemma 4.4. *There exist* 3 *idempotent* MOLS(n) *for* $n = 44$ *and* 52.

Combining Lemma 4.1, Lemma 4.2 and Lemma 4.4, we obtain the following result.

Theorem 4.5. *There exist* 3 *idempotent* MOLS(n) *for any positive integer* $n \notin \{2, 3, 4, 6, 10, 18, 22, 26, 28\}$.

It becomes easy to prove the following main theorem of this section.

Theorem 4.6. *There exists an* $(n, 5, 1)$-PMD *for any positive integer* $n \equiv 0 \pmod{10}$, *where* $n \neq 10, 20, 30, 50, 90, 110, 130, 140$.

Proof. From Theorem 4.5 we have 3 HMOLS(n) of type 1^n where n is even and $n \neq 2, 4, 6, 10, 18, 22, 26, 28$. Therefore, the conclusion follows from the $5n$-construction of Corollary 3.5. □

5. Construction of $(n, 5, 1)$-PMDs, $n \equiv 6 \pmod{10}$

We assume that the reader is familiar with the terminology of pairwise balanced designs (PBDs), group divisible designs (GDDs), and transversal designs (TDs) (see, for example, [5, 22]).

The following construction is [16, Lemma 2.1].

Lemma 5.1. *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a* GDD, *and let* $w: X \to \mathbb{Z}^+ \cup \{0\}$. *Suppose there are* k HMOLS *of type* $w(A)$, *for every block* $A \in \mathcal{B}$. *Then there are* k HMOLS *of type* $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.

In applying Lemma 5.1, we need several input designs. We know 3 HMOLS of type $1^7 2^1$, $1^8 2^1$ and $1^9 2^1$ all exist (see, for example, [16]). The 3 HMOLS of type 1^7, 1^8 and 1^9 come from Theorem 4.5.

Lemma 5.2. *Suppose there exists a* TD$(9, m)$. *Suppose there exist* 3 HMOLS *of type* 1^m *and* 1^{m-1}. *Then there exist* 3 HMOLS *of type* $8^{m-1}3^1(2u)^1$ *where* $0 \leq u \leq m-1$.

Proof. Deleting one point from a TD$(9, m)$, we obtain a GDD$[\{9, m\}, 1, \{8, m-1\}; 9m-1]$ of type $8^m(m-1)^1$. Further delete five points in some group of size eight, we obtain a new GDD with group type $8^{m-1}3^1(m-1)^1$ having block sizes $8, 9, m$, and $m-1$, where all blocks touching the group of size $m-1$ have sizes 8 and 9. Giving weight two to u points of the group of size $m-1$, weight zero to other points of the group, and weight one to the remaining points of the GDD, we apply Lemma 5.1 to get the required result. □

Lemma 5.3. *Suppose there exists a* TD$(12, m)$. *Then there exist* 3 HMOLS *of type* $(m-1)^8 3^1 (2u)^1$ *where* $0 \leq u \leq m-1$.

Proof. Take three points in one block of a TD$(12, m)$, and delete other points in the block. Further delete all other points in those three groups touching the three points. We obtain a GDD with group type $(m-1)^9 3^1$ and block sizes $10, 9, 8$. Give weight two to u points of some group of size $m-1$, weight zero to other points of the group, and weight one to the remaining points of the GDD. Applying Lemma 5.1 with the known input designs, we obtain the required result. □

Lemma 5.4. *If* $n \equiv 6 \pmod{10}$ *and* $n \geq 296$, *then there exists an* $(n, 5, 1)$-PMD.

Proof. It is well known (see, e.g., [5]) that a TD$(9, m)$ exists if $m \geq 781$. Apply Lemma 5.2 with the input 3 HMOLS from Theorem 4.5, we obtain 3 HMOLS of type $8^{m-1}3^1(2u)^1$ where $0 \leq u \leq m-1$. Using the $5n+1$-construction of Corollary 3.6, we obtain an $(n, 5, 1)$-PMD where $n = 10(4m+u) - 24$, $0 \leq u \leq m-1$. The required input designs $(v, 5, 1)$-PMDs for $v = 16, 41$ and $10u+1$ come from Theorem 1.2 and Theorem 2.1. This implies the existence of an $(n, 5, 1)$-PMD for $n \equiv 6 \pmod{10}$ and $n \geq 10(4.781 + 0) - 24 = 31216$.

Next, we apply Lemma 5.3 with odd prime power m to obtain 3 HMOLS of type $(m-1)^8 3^1 (2u)^1$ and an $(n, 5, 1)$-PMD for $n = 10(4m+u) - 24$, $0 \leq u \leq m-1$. We list the parameters in Table 1. Since $3264 > 4 \cdot 781$, we then have the existence of an $(n, 5, 1)$-PMD for $n \equiv 6 \pmod{10}$ and $n \geq 10(4 \cdot 37 + 0) - 24 = 1456$.

We further use Lemma 5.2 and Lemma 5.3 for small prime power m to get $(n, 5, 1)$-PMD where $n = 10(4m+u) - 24$, $0 \leq u \leq m-1$. The parameters are

Table 1

m	$4m$	$\leqslant 4m+u \leqslant$	$5m-1$
37	148		184
43	172		214
53	212		264
61	244		304
73	292		364
89	356		444
109	436		544
131	524		654
163	652		814
199	796		994
241	964		1204
293	1172		1464
359	1436		1794
443	1772		2214
523	2092		2614
653	2612		3264

listed in Table 2. Since $154 > 4 \cdot 37$, we then have the existence of an $(n, 5, 1)$-PMD for $n \equiv 6 \pmod{10}$ and $n \geqslant 10(4 \cdot 8 + 0) - 24 = 296$. The proof is now complete. □

To deal with the small values of n, we need more constructions of HMOLS and some special constructions for PMD. The following construction is adapted from [16, Corollary 3.3].

Lemma 5.5. *Suppose there exists a* $\mathrm{TD}(k+1, t)$, $\mathrm{TD}(k, m)$, *and* $\mathrm{TD}(k, m+1)$, *and* $0 \leqslant u \leqslant t-1$. *Then there exist* $k-2$ HMOLS *of type* $m^t u^1$.

Lemma 5.6. *There exists an* $(n, 5, 1)$-PMD *for* $n = 116$, 156, 176, 196, 216, 236, 256 *and* 276.

Table 2

m	$4m$	$\leqslant 4m+u \leqslant$	$5m-1$	Authority
8	32		39	Lemma 5.2
9	36		44	Lemma 5.2
11	44		54	Lemma 5.3
13	52		64	Lemma 5.3
16	64		79	Lemma 5.2
19	76		94	Lemma 5.3
23	92		114	Lemma 5.3
25	100		124	Lemma 5.3
31	124		154	Lemma 5.3

Proof. For $n=216$, we apply Lemma 5.5 with $m=8$, $k=t=5$ and $u=3$, and obtain 3 HMOLS of type $8^5 3^1$. For the remaining cases, applying Lemma 5.5 with $k=5$, $m=4$, $u=3$ and $t=5$, 7, 8, 9, 11, 12, 13, we obtain 3 HMOLS of type $4^t 3^1$. The conclusion then follows from the $5n+1$-construction.

Lemma 5.7. *There exists an* $(n, 5, 1)$-PMD *for* $n=76$, 106, 136 *and* 226.

Proof. It has been proved in [13] that 3 HMOLS of type 3^k exist for $k=5$, 7 and 9. This implies by the $5n+1$-construction that an $(n, 5, 1)$-PMD exits for $n=76$, 106 and 136. In addition, we also have a GDD of group-type 3^{15} and block size 7 due to Baker [1], which implies the existence of 3 HMOLS of type 3^{15}, and hence the existence of a $(226, 5, 1)$-PMD. □

Lemma 5.8. *There exists a* $(266, 5, 1)$-PMD.

Proof. We make the product construction of 3 MOLS(5) and 3 MOLS(10) with a hole of size two (see, e.g., [6]). Take four disjoint common transversals of the 3 MOLS(5), one of which is the main diagonal. After the product is done, replace the size 10 subarrays on the main diagonal by 3 HMOLS of type 2^5 and replace the subarrays on each of the remaining common transversals by some 3 MOLS which have two holes of size two and size one. After filling in those size two holes with 3 HMOLS of type 2^5, we obtain 3 HMOLS of type $2^{25}3^1$. Using the $5n+1$-construction we obtain the required $(266, 5, 1)$-PMD. □

Lemma 5.9. *There exists a* $(96, 5, 1)$-PMD.

Proof. Delete one point from a TD(5, 5), we obtain a GDD[5, 1, 4; 24]. Give weight 4 to each point and use the input design TD(5, 4). We get a GDD[5, 1, 16; 96]. Construct a $(5, 5, 1)$-PMD on each block and a $(16, 5, 1)$-PMD on each group, we obtain the required $(96, 5, 1)$-PMD.

Lemma 5.10. *Suppose there exist* $k-2$ MOLS(m). *Then*:

(1) *there exists an* $(mn, k, 1)$-HPMD *of type* $(mn_1, mn_2, \ldots, mn_h)$ *if there exists an* $(n, k, 1)$-HPMD *of type* $(n_1, n_2, \ldots, n_h)$; and

(2) *there exists an* $(mn, k, 1)$-HPMD *of type* m^n *if there exists an* $(n, k, 1)$-PMD.

Proof. It is obvious that (2) is a special case of (1) when the $(n, k, 1)$-HPMD is of type 1^n. To prove (1) we suppose $(X, \mathscr{C})$ is an $(n, k, 1)$-HPMD with holes $X_1, X_2, \ldots, X_h$ such that $|X_i| = n_i$, $1 \leq i \leq h$. Suppose the $k-2$ MOLS(m) are based on the set Y and corresponding to the quasigroups $(Y, \circ_i)$, $1 \leq i \leq k-2$. For each cycle $c=(c_1, c_2, \ldots, c_k) \in \mathscr{C}$ and any two elements x and y in Y (x may be equal to y), construct a cycle $((c_1, x\circ_1 y), (c_2, x\circ_2 y), \ldots, (c_{k-2}, x\circ_{k-2} y), (c_{k-1}, x), (c_k, y))$. All these cycles form the required HPMD with holes $X_1 \times Y$,

$X_2 \times Y, \ldots, X_h \times Y$. In fact, for any two elements (s, a) and (t, b) belonging to different holes $X_i \times Y$ and $X_j \times Y$ and for any $1 \leqslant r \leqslant k-1$, there is a cycle $(c_1, c_2, \ldots, c_k)$ such that $c_1 = s$ and $c_{1+r} = t$. Then the following equations

$$x \circ_1 y = a, \qquad x \circ_{1+r} y = b,$$

guarantee the existence of the elements x and y in Y since the quasigroups $(Y, \circ_1)$ and $(Y, \circ_{1+r})$ are orthogonal. So, we find the cycle along which the distance from (s, a) to (t, b) is r. On the other hand, suppose there is another cycle having the same property, say $((d_1, x_1 \circ_1 y_1), \ldots, (d_{k-2}, x_1 \circ_{k-2} y_1), (d_{k-1}, x_1), (d_k, y_1))$ where

$$(d_1, x_1 \circ_1 y_1) = (s, a), \qquad (d_{1+r}, x_1 \circ_{1+r} y_1) = (t, b).$$

From the definition of $(n, k, 1)$-HPMD and $(c_1, c_{1+r}) = (d_1, d_{1+r})$, we know that $(c_1, c_2, \ldots, c_k) = (d_1, d_2, \ldots, d_k)$. From the orthogonality of $(Y, \circ_1)$ and $(Y, \circ_{1+r})$ we also know that $(x_1, y_1) = (x, y)$. Therefore, the proof is complete. □

Lemma 5.11. *There exists a* $(166, 5, 1)$-PMD.

Proof. Applying Lemma 5.10(2) with $m = 15$, $n = 11$ and $k = 5$ we obtain a $(165, 5, 1)$-HPMD of type 15^{11}. Using Lemma 3.1 and Lemma 3.2 with a $(16, 5, 1)$-PMD, we have the required $(166, 5, 1)$-PMD. □

Lemma 5.12. *There does not exist any* $(6, 5, 1)$-PMD.

Proof. It is known from the proof of Theorem 5 in [3] that the existence of an $(n, k, 1)$-PMD implies the existence of a TD(k, n). Therefore, a $(6, 5, 1)$-PMD cannot exist since a TD$(4, 6)$ and then a TD$(5, 6)$ do not exist (see, for example, [15]). □

Combining Lemma 5.4, Lemmas 5.6–5.9, and Lemmas 5.11–5.12, we obtain the main result of this section.

Theorem 5.13. *There exists an* $(n, 5, 1)$-PMD *for any* $n \equiv 6 \pmod{10}$ *and* $n \geqslant 6$ *with the exception of* $n = 6$ *and the possible exception of* $n = 26$, 36, 46, 56, 66, 86, 126, 146, 186, 206, 246 *and* 286.

6. Conclusion

Combining Theorem 2.1, Theorem 4.6 and Theorem 5.13, we obtain the main result of this paper.

Theorem 6.1. *The necessary condition for the existence of an* $(n, 5, 1)$*-PMD, namely,* $n \equiv 0$ *or* $1 \pmod 5$ *and* $n \geq 5$*, is also sufficient with the exception of* $n = 6$ *and the possible exception of* $n = 10, 15, 20, 26, 30, 36, 46, 50, 56, 66, 86, 90, 110, 126, 130, 140, 146, 186, 206, 246$ *and* 286.

As another application of the kn and $kn+1$-constructions, we briefly mention the case $k = 7$ here. Since an $(n, 7, 1)$-PMD exists for $n = 7, 8$ from Theorem 1.1 and Theorem 1.2, and 5 HMOLS(m) of type 1^m exist for $m \geq 77$ from the list of 6 MOLS(m) contained in [7], we then apply Corollary 3.4 to obtain the following.

Theorem 6.2. *For any integer* $n \geq 539$ *where* $n \equiv 0$ *or* $1 \pmod 7$*, there exists an* $(n, 7, 1)$*-PMD.*

For the more general types of perfect Mendelsohn designs, further results on (n, k, λ)-PMDs will be reported in subsequent papers.

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Note added in proof. Recently, R.J.R. Abel has established the existence of 4 MOLS(28) which eliminate 28 from Lemma 4.1 and Theorem 4.5. Consequently, the number 140 can be deleted from the statement of Theorems 4.6 and 6.1. Also, E.R Lamken has constructed 3 HMOLS(57) of type 3^{19} and Corollary 3.6 can be applied to obtain a $(286, 5, 1)$-PMD. Thus 286 can be deleted from the statement of Theorems 5.13 and 6.1.

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